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Extending $\overline{\partial}$ to singular Riemann surfaces

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Dedicated to I.M. Gelfand
on his 75th birthday

Abstract. *The $\overline{\partial}$ operator is defined on Riemann surfaces with nodes. If X is a compact family of Riemann surfaces given by a map of X into $\overline{M}_g$, the Deligne-Mumford compactification of the moduli space M_g, then the family of operators $\{\overline{\partial}_x\}_{x\in X}$ is a continuous family. We compute its determinant line bundle.*

0. INTRODUCTION

We first explain our motivation for studying $\overline{\partial}$ on Riemann surfaces with nodes. In the Polyakov model for the bosonic string the path integral leads to a certain measure μ_g (the Polyakov measure) on M_g, the moduli space of complex structures for a compact surface of genus g. The measure is succintly described in terms of the determinant line bundle for a family of elliptic operators [BM].

Namely, let N_g denote the universal curve over M_g. Each fibre is a Riemann surface Σ with canonical bundle K and operator $\overline{\partial}(p) : K^p \to K^p \otimes \Lambda^{0,1}$; thus $\{\overline{\partial}(p)\}_{m\in M_g}$ is an elliptic family parametrized by M_g. (1) Its determinant line bundle (for $p = 1$) is

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(1) Strictly speaking, the elliptic family is defined for a compact family X of Riemann surfaces given by a map of $X \to M_g$.

called the Hodge line bundle λ. There is an explicit isomorphism of λ^{13} with $\mathcal{K}$ the canonical line bundle of M_g. Moreover λ has a natural Hermitian structure because $C^\infty(K)$ does $(\langle s_1, s_2\rangle = \frac{1}{2i}\int_\Sigma s_1 \wedge \bar{s}_2)$ and points of λ lie in $C^\infty(K)$. Hence λ^{13} and $\mathcal{K}$ inherit Hermitian structures; the measure μ_g is the volume element in $\mathcal{K}\otimes\overline{\mathcal{K}}$ of this Hermitian structure.

In string theory as well as algebraic geometry, it is natural to consider the Deligne-Mumford compactification $\overline{M}_g$ of M_g. Riemann surfaces occur as strings moving, breaking into two strings, and recombining. A string can shrink to a point and expand again, giving Riemann surfaces with simple nodes. These singular surfaces lie in the boundary $\overline{M}_g - M_g$. Algebraic geometers, using the dualizing sheaf and Grothendieck-Riemann-Roch, extend λ to a line bundle $\tilde{\lambda}$ over $\overline{M}_g$. Physicists approach the boundary by elongating a handle to an infinitely long cylinder. In any event, $\tilde{\lambda}^{13}$ is not isomorphic to the canonical line bundle $\tilde{\mathcal{K}}$ of $\overline{M}_g$. In fact, $\tilde{\lambda}^{-13}\tilde{\mathcal{K}}$ has divisor $-2(\overline{M}_g - M_g)$, closely related to the existence of a tachyon and the fact that $\int_{M_g}\mu_g = \infty$.

We study the behaviour near the boundary differently. Since the partition function and the Polyakov measure come from determinants of $\bar{\partial}(p)$ and their associated Laplacians, it seemed natural to try to define $\bar{\partial}(p)$ on a Riemann surface with node and study the behaviour of the $\bar{\partial}$ family over $\overline{M}_g$. Our results are described in the next section; briefly for p an integer, $\bar{\partial}(p)$ exists for a nodal surface and is a continuous family parametrized by $\overline{M}_g$. Its determinant line bundle is $\tilde{\lambda}^{6p^2-6p+1}$. When $p \in Z + \frac{1}{2}, M_g$ must be replaced by its spin moduli space $\tilde{M}_g$ whose compactification $\overline{\tilde{M}}_g$ is a branched covering over $\overline{M}_g$. Again $\bar{\partial}(p)$ is continuous family.

We hope that our technical innovation will be useful. In our approach, $\overline{M}_g$ and $\overline{\tilde{M}}_g$ are natural parameter spaces for $\bar{\partial}$. For superstrings, ultimately one should integrate over $\overline{\tilde{M}}_g$ to get scattering amplitudes. Having $\bar{\partial}(\frac{1}{2})$ defined at the boundary may illuminate the question of finiteness of the integrals.

Finally, the Dirac operator $\bar{\partial}(\frac{1}{2})$ has an interesting property. Suppose we construct a surface Σ_t of genus g from one Σ of genus $g-1$ by adding a handle, as in section 1; and suppose we choose a spin structure whose restriction to the annulus is of type $\sqrt{\mathrm{d}z}$. Then $\bar{\partial}_t(\frac{1}{2})$ varies continuously as $t \to 0$. Moreover $\bar{\partial}_0(\frac{1}{2})$ is just $\bar{\partial}(\frac{1}{2})$ on Σ. It would appear that $\bar{\partial}(\frac{1}{2})$ is well defined and continuous as parametrized by a universal closed spin moduli space, in the sense of Friedan and Shenker [FS] and J. D. Cohn [C].

1. DESCRIPTION OF RESULTS

On a compact Riemann surface Σ (maybe disconnected), choose two disjoint disks described locally as $\{|z| < 1\}$ and $\{|w| < 1\}$. For $0 < |t| < 1$, delete $\{|z| \le |t|\}$ and $\{|w| \le |t|\}$, and identify the remaining annuli, using $zw = t$. This adds a handle to Σ (or else reduces the number of components), creating a new Riemann surface Σ_t.

As $t \to 0$, we get a singular surface Σ_0, envisioned as Σ with the two points $\{z = 0\}$ and $\{w = 0\}$ identified.

Let $\delta = |t|^{1/2}$. We think of Σ_t as the surface outside the two disks, together with the sets $\{\delta \le |z| < 1\}$ and $\{\delta \le |w| < 1\}$, and identifying $\{|z| = \delta\}$ with $\{|w| = \delta\}$ by means of $zw = t$; this defines a rather arbitrarily chosen «cut» on Σ_t.

Let $K = \Lambda^{1,0}$ be the bundle of one-forms of type (1, 0), the canonical line bundle, so that $\bar K = \Lambda^{0,1}$; then for integer p, there is a map $\bar\partial(p)$ (briefly, $\bar\partial$) from sections of K^p to sections of $K^p \times \overline{K}$. (Later, we also consider half-integer p.) There is a natural pairing between sections $\tilde s$ of $K^p \otimes \overline{K}$ and s of K^{1-p}, defined by

$$\langle \tilde s, s\rangle = \int \tilde s \wedge s,$$

where $\tilde s \wedge s$ is viewed as an element of $\Lambda^{1,1} \cong K \otimes \overline{K} \cong K^p \otimes \overline{K} \otimes K^{1-p}$. With respect to this pairing, the dual of $\bar\partial : C^\infty(K^p) \to C^\infty(K^p \otimes \overline{K})$ is $-\bar\partial : C^\infty(K^{1-p}) \to C^\infty(K^{1-p} \otimes \overline{K})$.

Near the «cut» on Σ_t, $zw = t$ implies that $dz/z = -dw/w$, so it is natural to use $(dz/z)^p$ and $(dw/w)^p$ as local trivializing sections of K^p in the annuli $\{\delta \le |z| < 1\}$ and $\{\delta \le |w| < 1\}$, and to represent local sections as $f(dz/z)^p \oplus g(dw/w)^p$. Then the above pairing between such a section and $\tilde f (dz/z)^{1-p} d\bar z \otimes \tilde g (dw/w)^{1-p} \otimes d\bar w$ is

$$\text{(0.1)} \qquad \int_{\delta \le |z| < 1} \tilde f f \frac{d\bar z \wedge dz}{z} + \int_{\delta \le |w| < 1} \tilde g g \frac{d\bar w \wedge dw}{w}.$$

In polar coordinates $z = re^{i\alpha}$, $\frac{d\bar z \wedge dz}{z} = 2ie^{-i\alpha} dr \wedge d\alpha$; thus it is natural to define Hilbert spaces on Σ_t using a measure μ which agrees in $\delta \le |z| < 1$ with $dr\,d\alpha$, and in $\delta \le |z| < 1$ with $ds\, d\beta$, where $w = se^{i\beta}$; away from the cut, the measure is given by a smooth positive density on Σ. [This measure is not smooth at the cut, but its density is Lipschitz, and this is good enough to deal with a first order differential operator such as $\bar\partial$.] Using such a measure, we define Hilbert spaces $H_t^{p,q}$ of sections $K^p \otimes \overline{K}^q(\Sigma_t)$ such that, for f supported in the z-annulus, the norm of $f\frac{dz}{z}(d\bar z)^q$ is $[\int |f|^2 d\mu]^{1/2}$, and with the corresponding norm in the w-annulus. For $|t| > 0$, these are equivalent to the usual Hilbert spaces.

The Cauchy-Riemann operator $\bar\partial : C^\infty(K^p(\Sigma_t)) \to C^\infty(K^p \otimes \overline{K}(\Sigma_t))$ induces a family of unbounded operators

$$\mathrm{D}_t : H_t^{p,0} \supset \mathrm{dom}(\mathrm{D}_t) \to H_t^{p,1}.$$

Our main result is that this family has, in a certain sense, a limit D_0 as $t \to 0$. The extended family of unbounded Fredholm operators is continuous in the graph norm, as explained in Section 1 below. This implies that it is equivalent to a continuous family of bounded Fredholm operators from the graph of D_0 to L^2. Hence the index varies continuosly as $t \to 0$; and a continuous determinant line bundle can be defined in a neighborhood of $t = 0$. We identify this bundle in Section 8.

For integer p, the domain of the limiting operator D_0 is a subspace of $H_0^{p,0}$, which can be described locally near the cut as those sections which look like $u\left(\frac{dz}{z}\right)^p$ in $\{0 < |z| < 1\}$, like $v\left(\frac{dw}{w}\right)^p$ in $\{0 < |z| < 1\}$, with $u, \partial u/\partial\overline{z}, v$, and $\partial v/\partial\overline{w}$ in $L^2(d\mu)$, and which satisfy a «matching condition» $u(0) = (-1)^p v(0)$. [These function values are well defined when the L^2 conditions are met.] The limiting operator is of «regular singular» type, having the usual singularity for derivatives represented in polar coordinates. [In (0.1), we might have chosen to use $d\overline{z}/\overline{z}$ rather thant $d\overline{z}$; this would lead to a more singular measure, and to a limiting operator of the degenerate type studied by Melrose [M].]

The kernel and cokernel of the limiting operator D_0 can be identified with certain meromorphic sections of $K^p(\Sigma)$ and $K^{1-p}(\Sigma)$. The kernel consists of sections of $K^p(\Sigma)$, holomorphic for $z \neq 0, w \neq 0$, with poles of order $\leq p$ at those points, and with leading terms $a_0 z^{-p}$ and $b_0 w^{-p}$ that satisfy

$$a_0 = (-1)^p b_0 .$$

To describe the cokernel in similar terms, we use the pairing between sections of $K^p \otimes \overline{K}$ with sections of K^{1-p} to induce a pairing between $H_0^{(p,1)}$ and $H_0^{(1-p,0}$ that identifies each with the dual of the other. With respect to that pairing, the adjoint of

$$D_0 : \mathrm{dom}(D_0) \to H_0^{(p,1)}$$

is

$$D_0' : \mathrm{dom}(D_0') \to H_0^{(1-p,1)}$$

where D_0' is again $-\overline{\partial}$, with domain consisting of sections s in $H_0^{(1-p,0)}$ with $\overline{\partial} s$ in $H_0^{(1-p,1)}$, and which at $z = 0, w = 0$ are represented as $u\left(\frac{dz}{z}\right)^{1-p} \otimes \left(\frac{dw}{w}\right)^{1-p}$ with $u(0) = (-1)^{1-p} v(0)$. [See Section 2 below.] Thus the cokernel is identified with the kernel of D_0', sections of $K^{1-p}(\Sigma)$ having poles of order $\leq 1-p$ and leading terms that match appropriately.

For $p = 1/2$, the case of the Dirac operator $\overline{\partial}(1/2)$, the bundle $K(\Sigma_t)$ has two types of square root in the annulus $\{\delta \leq |z| < 1, \delta \leq |w| < 1\}$. For one type, local sections near $z = 0, w = 0$ are represented as $u\sqrt{dz/z} \oplus v\sqrt{dw/w}$, and the analysis proceeds as above. In the other type, local sections are $u\sqrt{dz} \oplus v\sqrt{dw}$; the pairing of such a section with $\tilde{u}\sqrt{dz} \otimes d\overline{z} \oplus \tilde{v}\sqrt{dw} \otimes d\overline{w}$ is then

$$\int \tilde{u} u \, d\overline{z} \wedge dz + \int \tilde{v} v \, d\overline{w} \wedge dw. \tag{0.2}$$

For this type, (0.2) suggests the measure given by $r\,d\,r\,d\,\alpha$ and $s\,d\,s\,d\,\beta$; with this measure, we again obtain a limiting operator D_0, but with *no* matching condition. Moreover, this measure gives the standard Lebesgue class on the original nonsingular surface Σ, and the local sections $\sqrt{d\,z}$ and $\sqrt{d\,w}$ can be interpreted as local sections of a square root of $K(\Sigma)$. So in this case, the limiting operator D_0 is simply $\bar\partial$ on a square root of $K(\Sigma)$; the «link» of $z = 0$ with $w = 0$ dissolves in the limit as $t \to 0$. In Section 6, we interpret this result in terms of a spin-moduli compactification. Then we describe the determinant line bundle of the family $\bar\partial$ parameterized by $\overline{M}_g$ and identify the family index with Grothendieck-Riemann-Roch for the dualizing sheaf. We also describe the spin moduli case.

In Section 7, we use the continuity of the $\bar\partial$ family to reprove the well-known Riemann-Roch index formula $\operatorname{ind}(\bar\partial(p)) = (2p-1)(g-1)$ for the operator

$$\bar\partial_p : C^\infty(K^p(\Sigma(g))) \to C^\infty(K^p \otimes \overline{K}(\Sigma(g)))$$

on a Riemann surface $\Sigma(g)$ of genus g. In this approach, the index is computed by passing to a limit point in moduli space, and comparing the limiting operator to the corresponding operator on a surface of lower genus, thus computing the variation of the index with respect to the genus.

2. A THEOREM OF CORDES AND LABROUSSE

The domain of the operator D_t varies with the parameter t; this situation is covered by a theorem of Cordes and Labrousse [CL].

We are given Hilbert spaces H_1 and H_2, and a family of closed operators D^t : $H_1 \supset \operatorname{dom}(D_t) \to H_2$. Denote by G_t the graph of D_t, and by P_t the othogonal projection on G_t. We call the family D_t *graph-continuous* if P_t is norm-continuous.

THEOREM. *If* D_0 *is Fredholm and* P_t *is continuous at* $t = 0$ *then, for small* t, D_t *is Fredholm with* $\operatorname{ind}(D_t) = \operatorname{ind}(D_0)$.

Proof. For t small, P_t induces an isomorphism $p_t : G_0 \to G_t$. Denote by i_t the isomorphism from $\operatorname{dom}(D_t)$ to the graph $G_t \subset H_1 \otimes H_2$, and by π_2 the projection from $H_1 \otimes H_2$ onto the second factor. Then

$$D_t = [\pi_2 p_t][p_t^{-1} i_t].$$

The right-hand factor is an isomorphism of $\operatorname{dom}(D_t)$ onto G_0, and the left-hand factor is a norm-continuous family of operators from G_0 into H_2. Since D_0 is assumed Fredholm, $\pi_2 p_0$ is Fredholm; this persists for small t, and the index is constant, proving the theorem.

To verify graph-continuity of the family D_t, note that it is equivalent to the graph-continuity of each of the following:

$$D_t^*; \qquad \begin{bmatrix} I & -D_t^* \\ D_t & I \end{bmatrix}; \quad \begin{bmatrix} I & -D_t^* \\ D_t & I \end{bmatrix}^{-1}.$$

The last of these operators is graph-continuous if and only if it is continuous in norm. We verify this norm-continuity by constructing right parametrics for D_t and D_t^*.

3. PARAMETRICES NEAR THE CUT

In polar coordinates $z = re^{i\alpha}$,

$$\partial u/\partial \bar{z} = \frac{1}{2}e^{i\alpha}\left(\frac{\partial u}{\partial r} + \frac{i}{r}\frac{\partial u}{\partial \alpha}\right).$$

To construct a right parametrix for

$$\bar{\partial} : u\left(\frac{\mathrm{d}\,z}{z}\right)^p \to (\partial u/\partial \bar{z})\left(\frac{\mathrm{d}\,z}{z}\right)^p \mathrm{d}\,\bar{z},$$
$$v\left(\frac{\mathrm{d}\,w}{w}\right)^p \to (\partial v/\partial \overline{w})\left(\frac{\mathrm{d}\,w}{w}\right)^p \mathrm{d}\,\overline{w}$$

suppose that we are given

$$\text{(3.1)} \qquad \begin{aligned} &f(r,\alpha)\frac{1}{2}e^{i\alpha}\left(\frac{\mathrm{d}\,z}{z}\right)^p \mathrm{d}\,\bar{z}, \quad \delta < r < 1, \\ &g(s,\beta)\frac{1}{2}e^{i\beta}\left(\frac{\mathrm{d}\,w}{w}\right)^p \mathrm{d}\,\overline{w}, \quad \delta < r < 1, \end{aligned}$$

Then we need u and v with

$$\text{(3.2)} \qquad \frac{\partial u}{\partial r} + \frac{i}{r}\frac{\partial u}{\partial \alpha} = f, \qquad \frac{\partial v}{\partial s} + \frac{i}{s}\frac{\partial v}{\partial \beta} = g,$$

satisfying an appropriate matching condition at the cut $|z| = r = \delta, |w| = s = \delta$. From $zw = t$, we have $\mathrm{d}\,z/z = -\mathrm{d}\,w/w$ and $\alpha + \beta = \arg(t)$. Let $\theta = \arg(t)$, so $t = \delta^2 e^{i\theta}$. Then the matching condition at the cut is

$$\text{(3.3)} \qquad u(\delta e^{i\alpha}) = (-1)^p v(\delta e^{i\theta - i\alpha}).$$

A pair of functions u and v into the Sobolev space H^1 that satisfy this condition combine to form an element of H^1 in the union of the two annuli where u and v are defined.

To invert (3.2), we use Fourier expansions $f(r,\alpha) = \Sigma f_k(r)e^{ik\alpha}$, and similary for g, u, v. Then (3.2) and (3.3) give

$$\text{(3.2')} \qquad \frac{\partial}{\partial r}u_k - \frac{k}{r}u_k = f_k, \ \frac{\partial}{\partial s}v_k - \frac{k}{s}v_k = g_k$$

and

$$\text{(3.3')} \qquad u_k(\delta) = (-1)^p e^{-ik\theta} v_{-k}(\delta).$$

The solutions of (3.2') are

$$u_k(r) = r^k \left[\int_?^r \rho^{-k} f_k(\rho)\,\mathrm{d}\rho + c_k\right],$$

$$v^k(s) = s^k \left[\int_?^s \sigma^{-k} g_k(\sigma)\,\mathrm{d}\sigma + \mathrm{d}_k\right].$$

The choice of lower limit of integration is determined by our wish for a limits as $\delta \to 0$, in the spaces $L^2(\mathrm{d}r\mathrm{d}\alpha)$ and $L^2(\mathrm{d}s\mathrm{d}\beta)$. When $\delta = 0$, then $\int_\delta^r \rho^{-k} f_k \mathrm{d}\rho$ will not converge for all f_k in L^2 unless $k \le 0$; and r^k is not in L^2 unless $k \ge 0$. Thus we define

$$\text{(3.4)} \qquad \begin{aligned} u_k &= r^k \int_1^r \rho^{-k} f_k \mathrm{d}\rho, \qquad k > 0 \\ &= \int_\delta^r f_0 \mathrm{d}\rho, \qquad k = 0 \\ &= r^k \left[\int_\delta^r \rho^{-k} f_k \mathrm{d}\rho + c_k(t)\right], \qquad k < 0, \end{aligned}$$

and $u_k = 0$ for $r < \delta$. The definition of v_k is nearly the same; substitute s for r, σ for ρ, g for f, and $d_k(t)$ for $c_k(t)$. We will see that these definitions do in fact give operators bounded in L^2, with continuous limits as $t \to 0$. The definition for $k = 0$ is not forced by the considerations adduced, but will satisfy the matching conditions (3.3) for $k = 0$. The other matching conditions are satisfied with

$$\text{(3.5)} \qquad \begin{aligned} c_k(t) &= (-1)^{p+1} t^{-k} \int_\delta^1 \sigma^k g_{-k} \mathrm{d}\sigma, \\ \mathrm{d}_k(t) &= (-1)^{p+1} t^{-k} \int_\delta^1 \rho^k f_{-k} \mathrm{d}\rho, \qquad k = -1, -2, \ldots \end{aligned}$$

Given a section defined by a pair $f \oplus g$ as in (3.1), we define the parametrix

$$Q_t(f \oplus g) = u \oplus v$$

with u and v defined as in (3.4) and (3.5).

LEMMA 1. *Considered as a map of* $L^2(0 < |z| < \delta, 0 < |w| < \delta)$ *into itself,* Q_t *is compact, and converges in norm to the operator* Q_0 *defined by (3.4) with* $\delta = 0, t = 0, c_k(0) = 0, \mathrm{d}_k(0) = 0$.

Proof. We estimate the norm of an integral operator $\int K(r,\rho)f(\rho)\mathrm{d}\rho$ on $L^2(\mathrm{d}r)$ by $\sup_\rho \int |K(r,\rho)|\mathrm{d}r + \sup_r \int |K(r,\rho)|\mathrm{d}\rho$, using «Schur's test» [HS, p. 22]. This shows that the integral operator in the k^{th} eigenspace (ignoring for now the terms $c_k(t)$ and $\mathrm{d}_k(t)$) has norm $O((1+|k|)^{-1})$, uniformly in $\delta < 1$. Hence the direct sum of these operators over the eigenspaces converges in norm. Since each is a compact integral operator, so is the sum. One shows likewise that the integral operators for $t \neq 0$ converge in norm, as $t \to 0$, to those in (3.4) with $t = \delta = 0$.

As for the terms with c_k and d_k, the norm of $r^k c_k(t)$ in $L^2(\delta, 1)$ is $O(\delta \log \frac{1}{\delta}(1+|k|)^{-1})||g_k||$, and there is a like estimate for $s^k \mathrm{d}_k(t)$. This proves the Lemma. ∎

The next lemma shows what happens to the matching conditions (3.3) when $t = 0$.

LEMMA 2. *Suppose that* u *and* $\dfrac{\partial u}{\partial r} + \dfrac{i}{r}\dfrac{\partial u}{\partial \alpha}$ *are in* $L^2(0 < r < \epsilon; \mathrm{d}r\mathrm{d}\alpha)$. *Then all Fourier coefficients* $u_k(r)$ *are* $O(r^{1/2})$ *except for* $u_0(r)$, *and that one has a limit as* $r \to 0$.

Proof. If $\dfrac{\partial u}{\partial r} + \dfrac{i}{r}\dfrac{\partial u}{\partial \alpha} = f$, then the Fourier coefficients of u are

$$u_k(r) = r^{-k}\left[\int_\epsilon^r \rho^k f_k + c_k\right].$$

Hence $u_0(r) = c_0 + \int_\epsilon^r f_0 \to c_0 - \int_0^\epsilon f_0$. For $k > 0$, $\lim_{r\to 0} \int_\epsilon^r \rho^k f_k = -\int_0^\epsilon \rho^k f_k$ exists; since $u_k(r)$ is in L^2, and r^{-k} is not, it follows that $\int_0^\epsilon \rho^k f_k = c_k$, hence

$$|u_k(r)| = r^{-k}\left|\int_0^r \rho^k f_k\right| = ||f_k|| \cdot O(r^{1/2}).$$

For $k < 0$ we have $\int_\epsilon^r \rho^k f_k = ||f_k|| \cdot O(r^{k+1/2})$, and the Lemma follows. ∎

Thus in the domain of D_0, $u(re^{i\alpha}) = \sum_{-\infty}^{\infty} u_k(r)e^{ik\alpha}$ with $u_k(0) = 0$ for $k \neq 0$, and $u_0(0) = u(0)$ well defined. The matching conditions in (3.3') are automatic for $k \neq 0$, and for $k = 0$ they are expressed by $u(0) = (-1)^p v(0)$.

As a consequence, we determine the domain of the adjoint D_0^*. Near the cut, let $\tilde{u} \oplus \tilde{v}$ stand for a pair of functions defining an element in $\mathrm{dom}(\mathrm{D}_0)$, and $u \oplus v$ be a pair in the domain of the adjoint. By the Lemma, these functions have boundary values $u(0)$ and $v(0)$ at the cut. Denote by D_0' the formal adjoint of D_0. Then

$$\begin{aligned}(D_0(\tilde{u}\oplus\tilde{v}), u\oplus v) &= (\tilde{u}\oplus\tilde{v}, D_0'(u\oplus v)) - \\ &\quad - 2\pi[\tilde{u}(0)\overline{u}(0) + \tilde{v}(0)\bar{v}(0)] \\ &= (\tilde{u}\oplus\tilde{v}, D_0'(\tilde{u}\oplus\tilde{v})) - 2\pi\tilde{u}(0)[\bar{u}(0) \\ &\quad + (-1)^p\bar{v}(0)]\end{aligned}$$

because of the matching condition on $\tilde{u}\oplus\tilde{v}$. But $\tilde{u}(0)$ can be chosen arbitrarly, so the domain of the adjoint requires that

$$u(0) = -(-1)^p v(0). \tag{3.6}$$

4. THE GLOBAL PARAMETRIX, AND GRAPH-CONTINUITY

Let D_t be the unbounded operator induced by $\bar{\partial}$ on $H_t^{p,0}$ as above. We will construct $\begin{bmatrix} I & -D_t^* \\ D_t & I \end{bmatrix}^{-1}$, and show the appropriate continuity as $t \to 0$.

Let Q_i (the «interior» parametrix) denote the usual pseudodifferential parametrix for $\bar{\partial}$ on the nonsingular compact Riemann surface Σ. Thus

$$\bar{\partial} Q_i f = f + S_i f;$$

the kernel of S_i is C^∞, and the kernel of Q_i is C^∞ off the diagonal. Near the cut, we use the parametrix Q_t constructed above; thus, set

$$Q = \varphi_i Q_i \psi_i + \varphi_c Q_t \psi_c$$

where $\psi_i + \psi_c = 1$ on $\Sigma_t, \varphi_i\psi_i = \psi_i, \varphi_c\psi_c = \psi_c$, φ_i and ψ_i vanish for $|z| < \frac{1}{2}$ and $|w| < \frac{1}{2}$, while φ_c and ψ_c are supported in $\{|z| < 1\} \cup \{|w| < 1\}$. We have

$$\begin{aligned} D_t Q &= [\varphi_i D_t Q_i \psi_i + \varphi_i' Q_i \psi_i] + [\varphi_c D_t Q_t \psi_c + \varphi_c' Q_t \psi_c] \\ &= I + R\end{aligned}$$

where $\varphi' = \bar{\partial}\varphi$, and the remainder is

$$R = \varphi_i S_i \psi_i + \varphi_i' Q_i \psi_i + \varphi_c' Q_c \psi_c.$$

By Lemma 1, R is a compact operator. Define Q' analogously as a right parametrix for D_t^*. Then

$$\begin{bmatrix} I & -\mathrm{D}_t^* \\ \mathrm{D} & I \end{bmatrix} \begin{bmatrix} 0 & Q \\ -Q' & 0 \end{bmatrix} = \begin{bmatrix} I+R' & Q \\ -Q' & I+R \end{bmatrix} \tag{4.1}$$

or $\tilde{\mathrm{D}}_t \tilde{Q}_t = \tilde{I} + \tilde{R}_t$, with $\tilde{R}_t$ compact, so $\tilde{I} + \tilde{R}_t$ is Fredholm with index 0. By its form, $\tilde{\mathrm{D}}_t$ is invertible, so $\tilde{Q}_t$ is a Fredholm operator from $\tilde{H}_t$ to $\mathrm{Dom}(\tilde{\mathrm{D}}_t)$, with index 0. Let N_0 be the nullspace of Q_0, and $k = \dim(N_0)$. Then $\mathrm{Range}(\tilde{Q}_0)$ has codimension k in $\mathrm{Dom}(\tilde{\mathrm{D}}_0)$, so there is a k-dimensional subspace V of $\mathrm{Dom}(\tilde{\mathrm{D}}_0)$ which is linearly independent of $\mathrm{Range}(\tilde{Q}_0)$. Let $\tilde{S}_0$ map N_0 isomorphically onto that k-dimensional subspace V, and map $N_0^{\perp}$ to 0. Then $\tilde{Q}_0 + \tilde{S}_0$ is an isomorphism of $\tilde{H}_0(\Sigma_0)$ to $\mathrm{Dom}(\tilde{\mathrm{D}}_0)$, so $\tilde{\mathrm{D}}_0(\tilde{Q}_0 + \tilde{S}_0) = \tilde{I} + \tilde{R}_0 + \tilde{\mathrm{D}}_0\tilde{S}_0$ is invertible, and

$$\tilde{\mathrm{D}}_0^{-1} = (\tilde{Q}_0 + \tilde{S}_0)\,(\tilde{I} + \tilde{R}_0 + \tilde{\mathrm{D}}_0\tilde{S}_0)^{-1}.$$

We now transfer the operators $\tilde{\mathrm{D}}_t, \tilde{Q}_t$ and $\tilde{R}_t$ in (4.1) to families acting in the fixed space $\tilde{H}_0$ of sections on Σ_0, rather than the varying spaces $\tilde{H}_t^{p,1}$ of sections on Σ_t. Take a continuous family of diffeomorphisms χ_t of $\{0 < |z| < 1, 0 < |w| < 1\}$ onto $\{\delta < |z| < 1, \delta < |w| < 1\}, \delta = |t|^{1/2}$, with χ_0 the identity. These induce unitary operators U_t mapping $\tilde{H}_0$ onto $\tilde{H}_t$, with $U_0 = I$. Composed with the natural embedding $\tilde{H}_t \subset \tilde{H}_0$, the family U_t is strongly continuous. By Lemma 1, $\tilde{Q}_t$ can be viewed as a norm-continuous family of compact operators in $\tilde{H}_0$, mapping $\tilde{H}_t$ to itself and $\tilde{H}_t^{\perp}$ to 0. Hence $U_t^*\tilde{Q}_tU_t$ is norm-continuous at $t = 0$. For, $\tilde{Q}_0$ can be approximated by an operator F of finite rank, and then

$$U_t^*\tilde{Q}_tU_t - \tilde{Q}_0 = (U_t^*FU_t - F) + U_t^*(\tilde{Q}_tU_t - F)U_t + (F - \tilde{Q}_0). \tag{4.2}$$

Here U_t is viewed as an isometry into $\tilde{H}_0$, and U_t^* as a partial isometry of $\tilde{H}_0$ annihilating functions supported in $\{|z| < \delta\} \cup \{|w| < \delta\}$, and $U_t^* \to I$ strongly, as $t \to 0$. Hence in (4.2), the last two terms can be made small in norm by choice of F, and then the first made small by taking t near 0. Similarly, $U_t^*\tilde{R}_tU_t$ is a norm-continuous family.

Now, $U_t^*U_t = \tilde{I}$ and $U_tU_t^* =$ projection on functions supported in $|z| > \delta, |w| > \delta$, so by (4.1)

$$(U_t^*\tilde{\mathrm{D}}_tU_t)\,(U_t^*\tilde{Q}_tU_t + S_0) = \tilde{I} + U_t^*\tilde{R}_tU_t + U_t^*\tilde{\mathrm{D}}_tU_t\tilde{S}_0.$$

The right-hand side is norm-continuous, and invertible when $t = 0$. Hence $U_t^*\tilde{\mathrm{D}}_tU_t$ has a norm- continuous inverse, and it follows that D_t, transferred via χ_t to the fixed space $H_0^{(p,0)}$, is a graph-continuous Fredholm family.

5. THE CASE $p = 1/2$

The case $p = 1/2$ concerns the Dirac operator $\bar{\partial}(1/2)$, acting on a square root of the canonical bundle K. Consider an annulus $A = \{\delta < |z| < 1\}$. Any line bundle over A is trivial. In particular, K has the cross section $\mathrm{d}z$. If L is a spin bundle (i.e. a square root of K) then $L^2 \cong K$. If s is a nonvanishing holomorphic section of L, then $s^2 = f\,\mathrm{d}z$ for some nonvanishing holomorphic function f. It is not hard to see that there are exactly two inequivalent cases: either f has a holomorphic square root on A, or else zf has such a root. Thus, K has two different square roots; one of the roots has $\sqrt{\mathrm{d}z}$ as nonvanishing section, and the other has $\sqrt{\mathrm{d}z/z}$. Only in the first case is there an extension to a root of the canonical bundle on the disk $\{|z| < 1\}$.

So in the annulus obtained by linking $\{\delta \leq |z| < 1\}$ with $\{\delta \leq |w| < 1\}$, local sections of a spin bundle are either $u\sqrt{\mathrm{d}z/z} \oplus v\sqrt{\mathrm{d}w/w}$ or $u\sqrt{\mathrm{d}z} \oplus v\sqrt{\mathrm{d}w}$, with appropriate matches between u and v.

Now consider a surface of genus g, with 2^{2g} (equivalence classes of) line bundles L such that $L \otimes L \cong \Lambda^{1,0} = K$. Let $K^{1/2}(\Sigma_t)$ denote one of these bundles. On the annulus $\{\delta < |z| < 1\}$ this induces one of the two possible bundle as above. The case $\sqrt{\mathrm{d}z/z}$, which does not extend to the disk, works out exactly as with integer p. Because of the square root, there are two possible matching conditions, and each has a distinct limit as $t \to 0$.

The case of $\sqrt{\mathrm{d}z/z}$, which does extend to $\{|z| < 1\}$, is more interesting. Since $\mathrm{d}z = -tw^{-2}\mathrm{d}w = -\delta^2 e^{i\theta} w^{-2}\mathrm{d}w$, one of the two possible identifications gives the matching condition

$$\sqrt{\mathrm{d}z} = i\delta e^{i\theta/2} w^{-1}\sqrt{\mathrm{d}z}.$$

At the cut, $z = \delta e^{i\alpha}$, and $\sqrt{\mathrm{d}z} = ie^{i(\alpha-\theta/2)}\sqrt{\mathrm{d}w}$, so the matching condition between $u\sqrt{\mathrm{d}z}$ and $v\sqrt{\mathrm{d}w}$ is

$$u(\delta e^{i\alpha}) = -ie^{-i(\alpha-\theta/2)} v(\delta e^{i\theta - i\alpha}). \tag{5.1}$$

Notice the monodromy; if $t = \delta^2 e^{i\theta}$ traces a circle around $t = 0$, the sign in (5.1) is reversed. Thus for each $t \neq 0$ there are two different line bundles which are exchanged by a circuit of $t = 0$; but in the limit as $t \to 0$, there will be just one bundle.

When we express the matching condition (5.1) in terms of the Fourier coefficients, there is a shift in the subscript:

$$u_k(\delta) = -ie^{-i\theta/2 - ik\theta} v_{-k-1}(\delta). \tag{5.2}$$

These conditions pair the indices $\{k \geq 0\}$ with $\{k \leq -1\}$. Thus we define our parametrix by

$$u_k(r) = r^k \int_1^r \rho^{-k} f_k \mathrm{d}\rho, \qquad k \geq 0$$
(5.3)
$$= r^k \left[\int_\delta^r \rho^{-k} f_k \mathrm{d}\rho + c_k(t)\right] \qquad k \leq -1,$$

with the analogous definition for v_k, containing a constant of integration $d_k(t)$, satisfying

$$c_k(t) = -ie^{-i\theta/2 - ik\theta}\delta^{-2k-1}\int_1^\delta \sigma^{k+1} g_{-k-1} \mathrm{d}\sigma,$$
(5.4)
$$\mathrm{d}_k(t) = -ie^{-i\theta/2 - ik\theta}\delta^{-2k-1}\int_1^\delta \rho^{k+1} f_{-k-1} \mathrm{d}\rho$$

for $k \leq -1$ As noted in the introduction, in the present case it is natural to use the measure $r\mathrm{d}\,r\mathrm{d}\,\alpha$ and the norms

$$||u\sqrt{\mathrm{d}\,z}||^2 = \int_\delta^\epsilon |u|^2 r\mathrm{d}\,r\mathrm{d}\,\alpha, ||f||^2 = \int_\delta^\epsilon |f|^2 r\mathrm{d}\,r\mathrm{d}\,\alpha.$$

We find that $||r^k c_k(t)|| = O(\delta\log\frac{1}{\delta}|k|^{-1})||g_{-k-1}||$, with the corresponding estimate for $s^k\mathrm{d}_k(t)$, and both tend uniformly to 0 as $t \to 0$. The integral operators $\int K(r,\rho)f(\rho)\,\mathrm{d}\rho$ acting in $L^2(r\mathrm{d}\,r)$ are estimated by $\sup_r \int |rK/\rho|\mathrm{d}\rho + \sup_\rho \int |K|\mathrm{d}\,r$ [HS, p. 22]; they have the obvious limits as $t \to 0$, as before.

We need also a parametrix for the adjoint of $\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\alpha}$ acting in $L^2(r\mathrm{d}\,r\mathrm{d}\,\alpha)$, with the matching conditions; the operator is

$$\mathrm{D}^* = -\frac{\partial}{\partial r} + r^{-1}(i\frac{\partial}{\partial\alpha} - 1),$$

and the matching condition is the same except for sign. Thus $\mathrm{D}^* u = f$ means $u_k' + r^{-1}(k+1)u_k = -f_k$. This leads to solutions as in (5.3), but with k replaced by $-k-1$; so once again, the desired limits exist. But *no matching condition survives*, and we are using the density $r\mathrm{d}\,r\mathrm{d}\,\alpha$, which is the natural density on the nonsingular surface Σ. (An elliptic first-order operator on a smooth compact manifold has only one closed realization, with domain the Sobolev space H^1. This precludes any matching conditions). Thus, in this case, the link between the annuli dissolves completely as $t \to 0$.

The case $p = 3/2$ is that of the Rarita Schwinger operator in superstring theory. As with $p = 1/2$, there are two distinct structures in the annulus. One has the nonvanishing section $\sqrt{\mathrm{d}\,z}\frac{\mathrm{d}z}{z}$, which is equivalent to $-i\delta e^{i\theta/2}w^{-1}\sqrt{\mathrm{d}\,w}\frac{\mathrm{d}\,w}{w}$. The corresponding matching condition is

$$u(\delta e^{i\alpha}) = ie^{-i\alpha + i\theta/2}v(\delta e^{i\theta - i\alpha}).$$

For the Fourier coefficients, this gives

$$u_k(\delta) = ie^{-i\theta(k+1/2)}v_{-k-1}(\delta).$$

This is the same as for $p = 1/2$, except for sign, so the parametrix is also the same. So the limiting operator, as for $p = 1/2$, has no matching condition, and the norm is

$$\|f\sqrt{\mathrm{d}z}\frac{\mathrm{d}z}{z}\mathrm{d}\bar z\|^2 = \int |f|^2 r\,\mathrm{d}r\,r\,\mathrm{d}\alpha.$$

The other type of line bundle of the annulus has nonvanishing section $\sqrt{\mathrm{d}z/z}\frac{\mathrm{d}z}{z} = (\mathrm{d}z/z)^{3/2}$, which is equivalent to $-i(\mathrm{d}w/w)^{3/2}$. Just as in the case of integer p [equation (3.3) above], the matching condition is

$$u(\delta e^{i\alpha}) = (-i)^{3/2}v(\delta e^{i\theta-i\alpha}) = \pm iv(\delta e^{i\theta-i\alpha}).$$

We get essentially the same matching conditions, and the same limit as $\delta \to 0$, with one surviving condition.

6. SPIN MODULI SPACE AND THE DETERMINANT LINE BUNDLE

Equivalence classes of Riemann surfaces of genus g with spin structure (choice of square root of the canonical bundle) form a branched 2^{2g} covering $\tilde M_g$ of moduli space. It has a natural compactification $\overline{\tilde M}_g$ giving a branched covering of $\overline{M}_g$. A functional description of $\overline{\tilde M}_g$ can be found in a letter from P. Deligne to Yu I. Manin, dated 25 September 1987. We describe briefly the relation of our $\bar\partial_0\left(\frac{1}{2}\right)$ to Deligne's compactification.

Let L be a line bundle giving a spin structure on the Riemann surface Σ_t. Then $L \otimes L \cong K$ or $L \cong \mathrm{Hom}(L, K) = K \otimes L^*$. Replace L and K by their sheaves of local sections $\mathcal{L}$ and $\mathcal{K}$ so that $\mathcal{L} \cong \mathrm{Hom}(\mathcal{L}, \mathcal{K})$. Now, following Deligne, on the singular surface Σ_0 we define a spin structure as a coherent torsion free analytic sheaf $\mathcal{F}$ so that $\mathcal{F} \cong \mathrm{Hom}(\mathcal{F}, \mathcal{K})$ where $\mathcal{K}$ is the sheaf of germs of the line bundle K_0 which is the canonical line bundle away from the node $z = 0 = w$ and whose non-vanishing section near the node is $\mathrm{d}z/z = -\mathrm{d}w/w$ for $(z, w) \neq (0, 0)$. (The domain of the limiting operator D_0 for $p = 1$ consists of sections of K_0 in the appropriate Sobolev class). Deligne points out, among other things, that $\overline{\tilde M}_g$ can be identified with the set of such sheaves $\mathcal{F}$.

Now when $\mathcal{F}$ is free, it gives a line bundle, a square root of K_0. When $\mathcal{F}$ is not free, it is given by a line bundle away from the node and at the node it is the direct sum of local sections of a line bundle on $0 \le |z| < \epsilon$ and a line bundle on $0 \le |w| < \epsilon$.

Our previous description of the domain of the limiting operator $\overline{\partial}_0\left(\frac{1}{2}\right)$ in section 5 is exactly an analytic description of $\mathcal{F}$ as the local solutions s to $\overline{\partial}_0\left(\frac{1}{2}\right)$. Thus $\overline{\partial}_0\left(\frac{1}{2}\right)$ is the element of the family $\{\overline{\partial}\left(\frac{1}{2}\right)\}_{\overline{\tilde{M}}}$ at $\mathcal{F} \in \overline{\tilde{M}}$.

The continuity of $\overline{\partial}(p)$ as $t \to 0$ for p an integer (section 4) shows (after allowing for a finite number of nodes) that $\{\overline{\partial}(p)\}_{m\in\overline{M}_g}$ is a continuous family of elliptic operators. The continuity proved in section 5 shows for $p \in Z + \frac{1}{2}$, that $\{\overline{\partial}(p)\}_{\mathcal{F}\in\overline{\tilde{M}}}$ is also a continuous family of elliptic operators.

Since $\{\overline{\partial}(p)\}_{\mathcal{F}\in\overline{M}}$ is a continuous family, it has a determinant line bundle DET_p over $\overline{M}_g$. Over M_g, the index theorem for families is the same as the Grothendieck-Riemann-Roch theorem. One obtains directly the Hodge line bundle $\lambda^{6p(p-1)+1}$ over M_g.

Now the canonical line bundle $\mathcal{K}$ of the fibers in N_g extends to a canonical line bundle $\overline{\mathcal{K}}$ over $\overline{N}_g$. Its sheaf of local holomorphic sections is the relative dualizing sheaf ω. Let $\pi : \overline{N}_g \to \overline{M}_g$ so that $\pi_!(\omega) \in K(\overline{M}_g)$. Let $\tilde{\lambda} = \det \pi_!(\omega)$; then $\tilde{\lambda}|_{M_g} = \lambda$. The matching condition for $p = 1$ in section 1 is the zero residue condition which characterizes $\omega[H]$. Hence $\ker \overline{\partial}_0 = H^0(\pi^{-1}(m), \omega)$ and $\ker(\overline{\partial}_0)^* = H^1(\pi^{-1}(m), \omega)$ so that $\mathrm{DET}_p = (\tilde{\lambda})^{6p(p-1)+1}$.

For $p = \frac{1}{2}$, $\mathrm{DET}\,\overline{\partial}\left(\frac{1}{2}\right)$ is a line bundle over $\overline{\tilde{M}}_g$. Presumbly it can be identified with $\det \tilde{\pi}_!(\tilde{\mathcal{F}})$ where $\tilde{\pi} : \overline{\tilde{N}}_g \to \overline{\tilde{M}}_g$ and $\tilde{\mathcal{F}}$ is the «Deligne sheaf» over $\overline{\tilde{N}}_g$ such that $\mathcal{F} = \mathrm{Hom}(\mathcal{F}, \omega)$.

It should be noted that because $H^{0,2}(\overline{\tilde{M}}_g) = 0$ and $H^{0,2}(\overline{M}_g) = 0$, we can conclude directly that the line bundle DET_p can be given a holomorphic structure.

7. THE INDEX

Here we sketch a proof of the Riemann-Roch index formula

$$\mathrm{ind}\,\overline{\partial}_g(p) = (2p-1)(g-1)$$

for $\overline{\partial}_g(p)$ acting on K^p of a Riemann surface $\Sigma(g)$ of genus g. When $g = 1, \Sigma(g)$ is a torus, and the index is trivially 0. We will show that the index drops by $2p-1$ when the genus drops by 1, hence the general formula. What is perhaps interesting in this proof is the direct comparison of indices by comparing boundary conditions at the node.

On $\Sigma(g)$, choose a generator of the fundamental group such that $\Sigma(g)$ can be identified with a surface Σ_1 constructed as in Section 1 from a surface Σ of genus $g-1$. By the continuity of the family $\mathrm{D}_t, \mathrm{ind}(\overline{\partial}_g(p)) = \mathrm{ind}(\mathrm{D}_0)$, where D_0 is the limiting operator «living» on Σ, with a domain allowing certain singularities at $z = 0$ and $w = 0$. We compare this to the usual $\overline{\partial}_{g-1}(p)$ on the nonsingular surface Σ, and show that the difference in the indices is $2p-1$.

To make the comparison, let

$$\Sigma' = \Sigma \backslash \{|z| \leq 1, |w| \leq 1\}.$$

Let $H^{[p,j]}$ be the Hilbertable space of L^2 sections of $K^p \otimes \overline{K}^j(\Sigma')$. In $H^{[p,0]}$ define $\overline{\partial}[p]$ with domain

$$\begin{aligned} \mathrm{D}^{[p,0]} = \{ & u \text{ in } H^{[p,0]} : \overline{\partial}_u \in H^{[p,1]}, \\ & u|_{|z|=1} = f\left(\frac{\mathrm{d}z}{z}\right)^p, u|_{|w|=1} = g\left(\frac{\mathrm{d}w}{w}\right)^p, \\ & f \text{ holomorphic in } |z| < 1, g \text{ holomorphic in } |w| < 1, \\ & f(0) = (-1)^p g(0)\}. \end{aligned} \tag{7.1}$$

The conditions on u along the two circles are a variant of the boundary conditions in [APS]. One can check that, with respect to the bilinear pairing between $H^{[p,1]}$ and $H^{[1-p,0]}$, the dual of $\overline{\partial}[p]$ is $-\overline{\partial}[1-p]$. Moreover, elements in the nullspace of $\overline{\partial}[p]$ have a natural and unique extension to elements in the nullspaces of our limiting operator D_0, as described in Section 1 above; thus these two nullspaces are isomorphic. The same is true for the nullspace of the dual operators, so the index of $\overline{\partial}[p]$ equals the index of D_0, which equals in turn the index of $\overline{\partial}_g(p)$.

We can construct an analogous operator from $H^{[p,0]}$ to $H^{[p,1]}$ whose kernel and cokernel are isomorphic to that of $\overline{\partial}_{g-1}(p)$ on Σ. The domain of this operator can be described as in (7.1), but now f and g have zeroes of order p at the origin, to cancel the poles in $(\mathrm{d}z/z)^p$ and $(\mathrm{d}w/w)^p$, and there is no matching condition $f(0) = (-1)^p g(0)$. This adds $2p-1$ conditions to the domain of $\overline{\partial}[p]$, thus reducing the index by $2p-1$, as claimed.

8. CONCLUDING REMARKS

If we have metrics on a family $X \subset M_g$ of Riemann surfaces, one can use determinants of $\overline{\partial}^* \overline{\partial}$ to put a Hermitian metric on DET $\overline{\partial}(p)|_X$ following Quillen [Q]. This gives a unique connection and curvature form expressing the Chern class of the line bundle explicitly as a 2-form. See [B] for example.

However, as one approaches the boundary $\overline{M}_g - M_g$, it is not clear what should replace metrics for a family containing a point in $\overline{M}_g - M_g$. Bismut and Bost [BB] put a metric on each non singular surface so as to give a smooth metric on the line bundle K_0 at the nodal surface. They then compute the 2-form representing the Chern class of the determinant line bundle of $\overline{\partial}(p)$, over M_g, which includes a Dirac delta function current with support on $\overline{M}_g - M_g$. Our procedure obtaining DET_p is different because our Laplacians are different. We hope to present a formula for the Chern class of DET_p and explain its relation to [BB] elsewhere.

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